Asymptotic properties of finite groups and finite dimensional algebras

Sept 19, 2008; Ottawa

Infinite Groups

Hopelessly infinite

Residually finite

$$G \xrightarrow{\varphi_i} G_i, \quad |G_i| < \infty, \quad \bigcap_i \ker \varphi_i = (1)$$

$\{\ker \varphi_i\}$ topology

Complete = profinite = inverse limit of finite groups

$G \hookrightarrow \hat{G}$ profinite completion

$G \to G/\cap\{H \,|\, |G:H| < \infty\} \to \hat{G}$

p prime, $G \xrightarrow{\varphi_i} G_i$, G_i finite

p-groups, $\bigcap_i \operatorname{Ker} \varphi_i = (1)$

G is residually-p

complete $=$ pro-p

$G \hookrightarrow G_{\hat{p}}$ pro-p completion

$G \to G/\cap\{H \,|\, |G:H| = p^k\} \to G_{\hat{p}}$

<u>EX. 1</u> F_m free group on $x_1, \ldots, x_m$

residually-p $\forall p$

$(F_m)_{\hat{p}}$ free pro-p group

Ex. 2 Λ commutative complete local Noetherian ring, $M \lhd \Lambda$

$\Lambda / M \cong GF(p^k)$

$GL'(n, \Lambda) = \{ A \in GL(n, \Lambda) \mid A \equiv I_n \bmod M \}$

Basic Problems

The Burnside Problem

(1) G finitely generated + torsion $\overset{?}{\Longrightarrow} |G| < \infty$

(2) G finitely generated + $\exists n : \forall g \in G \;\; g^n = 1 \overset{?}{\Longrightarrow} |G| < \infty$

Answers.

(1) No, Golod-Shafarevich (1964), Grigorchuk (1980)

(2) Generally speaking, No, Novikov-Adian (1968), but for Residually Finite Yes, E.Z. (1990)

Growth

$G = \langle X \rangle, \ |X| < \infty, \ 1 \in X = X^{-1}$

$X^n = \{ x_1 \cdots x_n, \ x_i \in X \}$

$X^1 \subset X^2 \subset \dots, \ \bigcup X^n = G$

$|X^1| < |X^2| < \dots$

J. Milnor: (1) Polynomial growth $\overset{?}{\Longleftrightarrow}$ $G \rhd H$, $|G:H| < \infty$, H nilpotent

(2) $\exists$? groups of intermediate growth

(1) 1980, Gromov: YES (new proof by Bruce Kleiner)

(2) 1980, Grigorchuk: YES

<u>The strongest form of fast growth: property τ</u>

$\Gamma = (V, E)$ finite connected graph; $W \subset V$, $\partial W = \{v \in V \mid dist(v, W) = 1\}$

Def. (Pinsker). $\varepsilon > 0$, Γ is an ε-expander if $|W| < \frac{1}{2}|V| \Rightarrow |W \cup \partial W| \geq (1+\varepsilon)|W|$

Wanted: families of finite K-regular ε-expanders; K, ε fixed; $|\Gamma| \to \infty$

Kazhdan Property (T) (1967): $G = \langle X \rangle$, $|X| < \infty$, $\exists \varepsilon > 0$: $\forall G \to U(H)$ unitary representation without $\neq 0$ fixed points, $\forall 0 \neq h \in H$ $\exists x \in X$: $\| h - xh \| \geq \varepsilon \cdot \|h\|$

(EX.: $G = SL(n, \mathbb{Z})$, $n \geq 3$)

G. A. Margulis (1982): G residually finite + Property (T), $G = \langle X \rangle$, $|X| < \infty$, $G \xrightarrow{\varphi_i} G_i$, $|G_i| < \infty$, $X \to X_i$. Then $\{ Cay(G_i, X_i) \}_i$ is an expander family.

A. Lubotzky - R. Zimmer: Property (τ)

Golod-Shafarevich examples.

$R \subseteq (F_m)_{\hat{p}} = F$, $N(R)$ = closed normal subgroup, generated by R

$F/N(R) = \langle x_1, \dots, x_m \mid R = 1 \rangle$

Thm (G-S, 1964) $R \subseteq F^p[F,F]$, $\# R < \frac{m^2}{4} \Rightarrow G$ is infinite

Thm (E.Z., 2000). A GS-group $\supset (F_m)_{\hat{p}}$

Ex. 1 S finite set of primes, $|S| = m$, $p \notin S$. Let $K/\mathbb{Q}$ = max pro-p extension, unramified outside of S. $Gal(K/\mathbb{Q}) = \langle x_1, \ldots, x_m \mid x_i^{p^{k_i}} = [x_i, a_i], 1 \le i \le m \rangle$ (I. Shafarevich, 1972)

Ex. 2 X = compact hyperbolic 3-manifold, $\Gamma = \pi_1(X)$. Then for almost all p $\Gamma \rhd H$, $|\Gamma : H| < \infty$, $H_{\hat{p}}$ is GS (A. Lubotzky, 1983).

$Gal(K/\mathbb{Q})$

Fontaine-Mazur Conj: $Gal(K/\mathbb{Q}) \to GL'(n, \mathbb{Q}_p)$ finite image

Weak form: $Gal(K/\mathbb{Q}) \not\hookrightarrow GL'(n, \Lambda)$

Thm $p >> n$, $\exists 1 \neq w(x_1, \dots, x_m) \in (F_m)_{\hat{p}}$:

$w(a_1, \dots, a_m) = 1 \quad \forall a_1, \dots, a_m \in GL'(n, \Lambda)$

COR. $Gal(K/\mathbb{Q})$, $(F_m)_{\hat{p}}$ not linear

$\pi_1(X)$

Conj. (Lubotzky-Sarnak).

$\pi_1(X)$ does not have (τ).

Virtual Positive Betti Number Conj.

(Thurston, Waldhausen, ...) :

$\pi_1(X) \rhd H$, $|\pi_1(X) : H| < \infty$, $H \twoheadrightarrow \mathbb{Z}$

M. Lackenby

<u>Conj.</u> (Lubotzky - Z., 1994-2006)

GS $\Rightarrow$ No τ

M. Ershov (to appear in Duke Math. J.)

$$d \geq 45 \,, \quad A = \underbrace{\begin{pmatrix} 2 & -1 & \cdots & -1 \\ -1 & \ddots & & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \cdots & & 2 \end{pmatrix}}_{d}$$

Kac - Moody group over $GF(p)$

$$p \geq \frac{1}{25} 1764^{d-1}$$

U = the subgroup, generated by positive root subgroups.

Carbone - Garland, Remy - Ronan, Dymara - Januszkiewich: U has (T).

Ershov (based on Tits's generators and relators): U is Golod-Shafarevich.

Very recently

<u>Ershov - Jaikin</u>: R a finitely generated associative ring,

$$E_n(R) = gp\langle I + e_{ij}(a) \mid 1 \le i \ne j \le n, a \in R \rangle$$

has property (T) if $n \geq 3$.

<u>Ershov - Jaikin</u>: G is GS $\Longrightarrow$

$G \underset{\text{finite}}{\rhd} H \twoheadrightarrow$ infinite group with (T).

<u>COR.</u> A GS group can not be amenable (conjectured by Vershik and de la Harpe)

<u>Potential COR.</u> Boston's strong form of the Fontaine-Mazur Conj.

Martin Kassabov: $S_n = \langle X_n \rangle$, $|X_n| \leq \text{Const}$, $n \to \infty$, $\{Cay(S_n, X_n)\}_n$ is an expander family.

Algebras

F a field, $A_F = \langle V \rangle$, $\dim V < \infty$

$a \in A$ is <u>algebraic</u> if $\exists f_a(t) \in F[t]$

$f_a(a) = 0$.

A is <u>algebraic</u> if every element

is algebraic.

A is <u>nil</u> if every element is

nilpotent.

<u>Kurosh Problem</u>: A f.g. + algebraic

$\overset{?}{\Longrightarrow} \dim_F A < \infty$

Golod - Shafarevich : NO

Growth

$A = \langle V \rangle$, $\dim_F V < \infty$

$V^n = \mathrm{Span}$ (products in V of length $\leq n$)

$V^1 \subset V^2 \subset \dots$, $\bigcup V^n = A$

$\dim V^1 < \dim V^2 < \dots$

$$\mathrm{GKdim}(A) = \overline{\lim_{n \to \infty}} \frac{\ln \dim V^n}{\ln n}$$

Gelfand-Kirillov dimension.

Problem. Is every nil algebra of polynomial growth finite dimensional?

For groups: Yes (Gromov)

T. Lenagan - A. Smoktunowicz (2007):

Such algebras exist over countable fields

Inductive limit of Golod-Shafarevich algebras, $\mathrm{GKdim}(A) \approx 30$

Another approach (Petrogradsky-Shestakov-Z.)

$\mathrm{char}\, F = p > 0$, $F[t_0, t_1, \dots]$

$\partial_i = \partial/\partial t_i$ $\qquad t_i^p$ constants

$F[t_0, t_1, \dots]/id(t_i^p, i \geq 0)$

$v_1 = \partial_1 + t_0^{p-1}\partial_2 + (t_0 t_1)^{p-1}\partial_3 + \cdots$

$v_2 = \partial_2 + t_1^{p-1}\partial_3 + (t_1 t_2)^{p-1}\partial_4 + \cdots$

$L = \mathrm{Lie}\langle v_1, v_2 \rangle$, $A = \mathrm{Assoc}\langle v_1, v_2 \rangle$

$\lambda = \dfrac{1+\sqrt{4p-3}}{2}$, $\quad \lambda^2 - \lambda - (p-1) = 0$

<u>Thm</u> (1) L is nil, $\forall a \in L$ $a^{p^n} = 0$

(2) $\mathrm{GKdim}(L) = \log_\lambda p$

(3) $\mathrm{GKdim}(A) = 2 \log_\lambda p$

<u>Conjecture</u>: A is nil

$U = U(L)$, $a^{p^n} \in Z$, center

$(Z \setminus \{0\})^{-1} U = D$

<u>Conjecture</u>: D is an algebraic division algebra

Fast Growth: Dimension Expanders

<u>Def.</u> $\varepsilon > 0$; $T_1, \dots, T_m \in \mathrm{End}_F(V)$ is an

<u>ε-dimension expander</u> if

(1) $T_1, \dots, T_m$ act irreducibly,

(2) $\forall\, W < V$, $\dim W < \frac{1}{2} \dim V$

$$\dim\Big(W + \sum_{i=1}^{m} T_i W\Big) \geq (1+\varepsilon) \dim W$$

<u>Problem.</u> (A. Wigderson). Find a family of ε-dimension expanders with m, ε fixed, $\dim V \to \infty$

<u>Def.</u> An algebra $A = \langle x_1, \dots, x_m \rangle$ has (τ) if $\exists\, \varepsilon > 0$: $\forall$ finite dim. irreducible ${}_A V$, $\forall\, (0) \neq W < V$, $\dim W < \frac{1}{2} \dim V$

Ex. 1 char $F = 0$, $F_p = \mathbb{Z}/p\mathbb{Z}$

$F[SL'(n, F_p[t])]$, $n \geq 3$, has (τ)

(A. Lubotzky - E. Z.)

Ex. 2 char $F = 0$, $\mathfrak{g}$ finite dimensional simple Lie algebra, $U(\mathfrak{g})$ has (τ)

Question: finite fields?